JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **195**, 148–159 (1995)

On a Third Order Nonlinear Boundary Value Problem at Resonance

R. KENT NAGLE AND KENNETH L. POTHOVEN

Department of Mathematics, University of South Florida, Tampa, Florida 33620-5700

Submitted by Zhivko S. Athanassov

Received January 15, 1993

In this paper we discuss the existence of solutions to third order boundary value problems of the form

$$x''' + x' + g(x, x') = p(t),$$
$$x'(0) = x'(\pi) = x(\eta) = 0,$$

where η lies in $[0, \pi]$. We assume that g is continuous and one-sided bounded (e.g., $g \geq 0$) and p is continuous. Solutions are shown to exist if p satisfies a Landesman–Lazer type condition involving g. Several examples are given to illustrate how these conditions depend upon the parameter η. © 1995 Academic Press, Inc.

1. INTRODUCTION

In this paper we consider boundary value problems of the form

$$\begin{aligned} x''' + x' + g(x, x') &= p(t), \\ x'(0) = x'(\pi) = x(\eta) &= 0, \end{aligned} \tag{1}$$

where $\eta \in [0, \pi]$. We assume that g is bounded on one side—either from below or from above.

In recent years there has been a renewed interest in third order three-point boundary value problems. Krajcinovac [9] studied the deflection of a three-layered elastic beam using a linear third order three-point boundary value problem and his work motivated Aftabizadeh and Xu [1] and Gupta [6] to study nonlinear versions of his model. Others who have recently

0022-247X/95 $12.00

studied the existence and uniqueness for third order boundary value problems are Aftabizadeh and Wiener [2], Henderson [7], Murty and Prasad [11, 12], and O'Regan [13]. Agarwal [3] studied existence and uniqueness in conjunction with iterative methods for approximating the solution. For background on linear third-order boundary value problems, see the book by Gregus [5].

Gupta [6] discussed the existence of solutions to boundary value problems similar to (1) of the type

$$x''' + \pi^2 x' + g(t, x, x', x'') = p(t),$$
$$x'(0) = x'(1) = x(\eta) = 0, \qquad 0 \leq \eta \leq 1,$$

under the conditions that

$$\int_0^1 p(t) \sin \pi t \, dt = 0;$$
$$g(t, u, v, w)v \geq 0 \qquad \text{for } t \in [0, 1],\ u, v, w \in R;$$

and

$$\limsup_{|v| \to \infty} \frac{g(t, u, v, w)}{v} < 3\pi^2,$$

uniformly for $(t, u, v) \in [0, 1] \times R^2$. Gupta uses a topological degree theory approach and a result of Mawhin [10, Theorem IV.4] to obtain his results.

In this paper, in consideration of (1), we do not require any of the three above conditions, but do require that g be one-sided bounded—either from above or from below. We also obtain results for a broader class of forcing functions p. Interestingly, these results depend on the parameter η.

2. The Main Result

To study (1), we reduce it to a second order problem by means of the transformation $x(t) = \int_\eta^t y(s)\, ds$. This gives the equivalent problem

$$y''(t) + y(t) + g\left(\int_\eta^t y(s)\, ds, y(t)\right) = p(t),$$
$$y(0) = y(\pi) = 0. \tag{2}$$

We will assume $g(u, v)$ is always nonnegative. The generalization to the other cases, to wit, g is bounded from below or g is bounded from above, is easily obtainable.

Let N be the Nemytskii operator on $C[0, \pi]$ generated by

$$N(y)(t) = p(t) - g\left(\int_\eta^t y(s)\, ds, y(t)\right).$$

Now N: $C[0, \pi] \to Z$, where Z is $C[0, \pi]$ with norm $\|f\|_z = \int_0^\pi |f(t)| \sin t\, dt$. Let P: $L^\infty[0, \pi] \to L^\infty[0, \pi]$ be the projection onto the subspace spanned by $\sin t$ given by

$$Pf = \frac{2}{\pi}\left(\int_0^\pi f(t) \sin t\, dt\right) \sin t.$$

For each $x \in L^\infty[0, \pi]$, let $x(t) = \rho \sin t + x_1(t)$, where $\int_0^\pi x_1(t) \sin t\, dt = 0$. For each $f \in C[0, \pi]$ such that $\int_0^\pi f(t) \sin t\, dt = 0$, let Hf represent the solution x of

$$x'' + x = f, \qquad x(0) = x(\pi) = 0,$$

with $\int_0^\pi x(t) \sin t\, dt = 0$. We then obtain the operator $H(I - P)$: $Z \to C[0, \pi]$ satisfying

$$\|H(I - P)f\|_\infty \leq \frac{2K}{\pi} \int_0^\pi |f(t)| \sin t\, dt = \frac{2K}{\pi} \|f\|_z,$$

for some constant K (see Kannan and Ortega [8]).

Problem (2) is equivalent to the equation

$$(I - P)x = H(I - P)N(x) + PN(x) \tag{3}$$

or

$$x_1 = H(I - P)Nx + PN(x),$$

where $x = \rho \sin t + x_1(t)$ for some ρ (see Cesari [4]). We now state the theorem:

THEOREM 1. *Assume*:

(i) g: $R \times R \to R$ *is a continuous function satisfying* $g(u, v) \geq 0$ *for all* (u, v) *in* $R \times R$.

(ii) $p \in C[0, \pi]$.

(iii) *There exist real numbers* $\rho_1 < \rho_2$ *and* $\delta > 0$ *such that*

$$\int_0^\pi g\left(\rho_1(\cos\eta - \cos t) + \int_\eta^t y_1(s)\,ds, \rho_1 \sin t + y_1(t)\right) \sin t\,dt$$
$$< p(t) \sin t\,dt$$
$$< \int_0^\pi g\left(\rho_2(\cos\eta - \cos t) + \int_\eta^t y_1(s)\,ds, \rho_2 \sin t + y_1(t)\right) \sin t\,dt$$

for $\|y_1\|_\infty < \delta$.

(iv) *Moreover,* δ *further satisfies*

$$\frac{4K}{\pi}\left[\int_0^\pi |p(t)| \sin t\,dt\right] < \delta.$$

Then, problem (1) *has a solution* $x(t) = \rho(\cos\eta - \cos t) + \int_\eta^t x_1(s)\,ds$, *where* $\|x_1\|_\infty < K[|p_0| + (2/\pi)\int_0^\pi |p(t)| \sin t\,dt]$ *and* ρ *is between* ρ_1 *and* ρ_2.

Proof. For $\lambda \in [0, 1]$, we consider the homotopic problems

$$x_1 = \lambda H(I - P)N(x) + \lambda PN(x) - (1 - \lambda)\varepsilon(\rho - \rho^*) \sin t, \tag{4}$$

where $\rho^* = (\rho_1 + \rho_2)/2$ and $\varepsilon > 0$ satisfies

$$K\left[\frac{4}{\pi}\int_0^\pi |p(t)| \sin t\,dt + \varepsilon\frac{|\rho_2 - \rho_1|}{2}\right] < \delta.$$

Let $R_0 > \delta$ and let Ω be given by

$$\Omega = \{x \in C[0, \pi] | x(t) = \rho \sin t + x_1(t), \|x_1\|_\infty < R_0,$$
$$\int_0^\pi x_1(t) \sin t\,dt = 0, \text{and } \rho \text{ is between } \rho_1 \text{ and } \rho_2\}.$$

Letting $C(\lambda)(x)$ be the compact operator given by

$$Px + \lambda H(I - P)N(x) + \lambda PN(x) - (1 - \lambda)\varepsilon(\rho - \rho^*) \sin t,$$

we use Leray–Schauder degree theory for the operator $I - C(\lambda)$ on the open, bounded set Ω.

If $\lambda = 0$, the mapping $I - C(\lambda)$ becomes

$$\begin{aligned}(I - C(0))(x) &= x - Px + \varepsilon(\rho - \rho^*)\sin t\\ &= x_1 + \varepsilon(\rho - \rho^*)\sin t.\end{aligned}$$

Solving $(I - C(0))(x) = 0$, we find

$$x_1 + \varepsilon(\rho - \rho^*)\sin t = 0$$

and so since x_1 is in the subspace orthogonal to span$\{\sin t\}$,

$$x_1 = 0 \qquad \text{and} \qquad \varepsilon(\rho - \rho^*)\sin t = 0.$$

Hence, $x_1 = 0$, $\rho = \rho^*$, and we have the unique solution $x = \rho^* \sin t$.

To compute the deg$(I - C(0), \Omega, 0)$, we first observe that $C(0)$ is a finite rank operator whose range lies in the span$\{\sin t\}$. Therefore, deg$(I - C(0), \Omega, 0) =$ deg$(I - C(0)|_{D_v}, D_v, 0)$, where $D_v = \Omega \cap$ span$\{\sin t\}$.

Now $(I - C(0))|_{D_v}(\rho \sin t) = \varepsilon(\rho - \rho^*)\sin t$ because $x_1 = (I - P)x$ lies in the subspace orthogonal to span$\{\sin t\}$. Hence $[I - C(0)]|_{D_v}$ is just a mapping from span$\{\sin t\}$ into span$\{\sin t\}$. We can identify the span$\{\sin t\}$ with $\mathbf{R}$ and consider the degree of the mapping $\varphi(\rho) = \varepsilon(\rho - \rho^*)$ from $[\rho_1, \rho_2]$ into $\mathbf{R}$. Since $\varphi(\rho) = 0$ has the unique solution $\rho = \rho^* = \frac{1}{2}(\rho_1 + \rho_2)$, which lies in $[\rho_1, \rho_2]$, and $\varphi'(\rho) = \varepsilon > 0$, the deg$(\varphi, [\rho_1, \rho_2], 0) = 1 \neq 0$. Consequently, deg$(I - C(0), \Omega, 0) = 1 \neq 0$.

If for $\lambda = 1$, $(I - C(1))(x) = 0$ has a solution, the theorem is proved. It remains to show then that for $0 < \lambda < 1$, $(I - C(\lambda))(x) = 0$ or (4) does not have a solution on the boundary of Ω.

Suppose that for $0 < \lambda < 1$, $x(t)$ is a solution. Then,

$$\lambda PN(x) - (1 - \lambda)\varepsilon(\rho - \rho^*)\sin t = 0,$$

which gives

$$\begin{aligned}\lambda \frac{2}{\pi}\int_0^\pi g\left(\int_\eta^t x(s)\,ds, x(t)\right)\sin t\,dt &= \lambda\frac{2}{\pi}\int_0^\pi p(t)\sin t\,dt - (1 - \lambda)\varepsilon(\rho - \rho^*)\\ &\le \frac{2}{\pi}\int_0^\pi |p(t)|\sin t\,dt + \varepsilon\frac{|\rho_2 - \rho_1|}{2}.\end{aligned}$$

Since $x_1 = \lambda H(I - P)N(x)$, we have

$$\begin{aligned}
\|x_1\|_\infty &= \lambda\|H(I-P)N(x)\|_\infty \leq \frac{\lambda 2K}{\pi}\|N(x)\|_z \\
&= \frac{\lambda 2K}{\pi}\int_0^\pi \left|-g\left(\int_\eta^t x(s)\,ds, x(t)\right) + p(t)\right| \sin t\,dt \\
&\leq \frac{\lambda 2K}{\pi}\left[\int_0^\pi g\left(\int_\eta^t x(s)\,ds, x(t)\right)\sin t\,dt + \int_0^\pi |p(t)|\sin t\,dt\right] \\
&\leq K\left[\frac{4}{\pi}\int_0^\pi |p(t)|\sin t\,dt + \varepsilon\frac{|\rho_2 - \rho_1|}{2}\right] \\
&< \delta < R_0.
\end{aligned}$$

Consequently, there is no solution for $\lambda \in (0, 1)$ on the boundary of Ω satisfying $\|x_1\|_\infty = R_0$.

Finally, we show there is no solution for ρ equal to ρ_1 or ρ_2. Suppose $x = \rho_2 \sin t + x_1(t)$ is a solution. From above $\|x_1(t)\|_\infty < \delta$. By our hypothesis (iii),

$$\begin{aligned}
-PN(x) = \frac{2}{\pi}\int_0^t \Bigg[g\Bigg(&\int_\eta^t \rho_2 \sin(s)\,ds + \int_\eta^t x_1(s)\,ds, \rho_2\sin(t) \\
&+ x_1(t)\Bigg)\sin t - p(t)\sin t\Bigg] dt \geq 0,
\end{aligned}$$

or

$$PN(x) \leq 0.$$

But $\lambda PN(x) = (1 - \lambda)\varepsilon(\rho_2 - \rho^*) \sin t > 0$ so that $\rho_2 \sin t + x_1(t)$ cannot be a solution.

Similarly $\rho_1 \sin t + x_1(t)$ cannot be a solution and the theorem is proved. Q.E.D.

Remark 1. If in the preceding theorem hypothesis (iii) is replaced by:

(iii)′ There exist real numbers $\rho_1 < \rho_2$ and $\delta > 0$ such that

$$\int_0^\pi g\left(\rho_1(\cos\eta - \cos t) + \int_\eta^t y_1(s)\,ds, \rho_1 \sin t + y_1(t)\right)\sin t\,dt$$
$$> \int_0^\pi p(t)\sin t\,dt$$
$$> \int_0^\pi g\left(\rho_2(\cos\eta - \cos t) + \int_\eta^t y_1(s)\,ds, \rho_2 \sin t + y_1(t)\right)\sin t\,dt$$

for $\|y_1\|_\infty < \delta$,
then the theorem follows with a similar proof with $\rho^* - \rho$ replacing $\rho - \rho^*$ in (4) and subsequent expressions.

3. Applications

Example 1. Consider the problem

$$\begin{aligned} x''' + x' + e^x + e^{x'} &= p(t), \\ x'(0) = x'(\pi) = x(\eta) &= 0, \end{aligned} \tag{5}$$

for $\eta \in [0, \pi]$.

If $\eta = 0$, then this problem has a solution if and only if $0 < \int_0^\pi p(t) \sin t\,dt < \infty$; but no solution is guaranteed by Theorem 1 if $\eta = 0$. Indeed, for $\rho \in R$, $\|y_1\|_\infty < \delta$, and $\int_0^\pi y_1(t)\sin t\,dt = 0$, consider

$$\Gamma(\rho, y_1) := \int_0^\pi g\left(\rho(\cos\eta - \cos t) + \int_\eta^t y_1(s)\,ds, \rho\sin t + y_1(t)\right)\sin t\,dt,$$

which for $g(x, x') = e^x + e^{x'}$ is

$$\begin{aligned}\Gamma(\rho, y_1) &= \int_0^\pi [e^{\rho(\cos\eta - \cos t)} e^{\int_\eta^t y_1(s)ds} + e^{\rho\sin t} e^{y_1(t)}]\sin t\,dt \\ &= \int_0^\eta [e^{\rho(\cos\eta - \cos t)} e^{\int_\eta^t y_1(s)ds} + e^{\rho\sin t} e^{y_1(t)}]\sin t\,dt \\ &\quad + \int_\eta^\pi [e^{\rho(\cos\eta - \cos t)} e^{\int_\eta^t y_1(s)ds} + e^{\rho\sin t} e^{y_1(t)}]\sin t\,dt.\end{aligned}$$

Now as $\rho \to \infty$, $\Gamma(\rho, y_1) \to \infty$ and as $\rho \to -\infty$, $\Gamma(\rho, y_1) \to \infty$ when $\eta > 0$ whereas $\Gamma(\rho, y_1) \to 0$ when $\eta = 0$. Consequently, if $\eta = 0$ and $0 < \int_0^\pi p(t) \sin t\,dt < \infty$, then δ, ρ_1, and ρ_2 exist such that $\rho_1 < \rho_2$ and

$$\Gamma(\rho_1, y_1) < \int_0^\pi p(t) \sin t \, dt < \Gamma(\rho_2, y_1)$$

provided $\|y_1\|_\infty < \delta$.

The necessity follows by multiplying Eq. (5) by $\sin t$, integrating from 0 to π, and substituting in upper and lower bounds on the integrand $0 < (e^x + e^{x'}) \sin t < \infty$.

EXAMPLE 2. Consider the problem

$$\begin{aligned} x''' + x' + e^{-x} + e^{x'} &= p(t), \\ x'(0) = x'(\pi) = x(\eta) &= 0, \end{aligned} \tag{6}$$

for $\eta \in [0, \pi]$.

If $\eta = \pi$, then this problem has a solution if and only if $0 < \int_0^\pi p(t) \sin t \, dt < \infty$ but no solution is guaranteed if $\eta < \pi$.

In this case we must consider

$$\begin{aligned} T(\rho, y_1) := &\int_0^\eta [e^{-\rho(\cos\eta - \cos t)} e^{-\int_\eta^t y_1(s)ds} + e^{\rho \sin t} e^{y_1(t)}] \sin t \, dt \\ &+ \int_\eta^\pi [e^{-\rho(\cos\eta - \cos t)} e^{-\int_\eta^t y_1(s)ds} + e^{\rho \sin t} e^{y_1(t)}] \sin t \, dt, \end{aligned}$$

for $\|y_1\|_\infty < \delta$. Now as $\rho \to \infty$, $T(\rho, y_1) \to \infty$ and as $\rho \to -\infty$, $T(\rho, y_1) \to \infty$ when $\eta < \pi$ whereas $T(\rho, y_1) \to 0$ when $\eta = \pi$.

EXAMPLE 3. Consider the problem

$$\begin{aligned} x''' + x' + e^x + \arctan(x') &= p(t), \\ x'(0) = x'(\pi) = x(\eta) &= 0, \end{aligned} \tag{7}$$

for $\eta \in [0, \pi]$.

If $\eta = 0$ (or π), problem (7) has a solution if $-\pi$ (or π) $< \int_0^\pi p(t) \sin t \, dt < \infty$, but no solution is guaranteed when $0 < \eta < \pi$. Moreover, it is only necessary that $-\pi < \int_0^\pi p(t) \sin t \, dt < \infty$.

In this case we must consider the equation

$$\begin{aligned} \sum (\rho, y_1) = &\int_0^\eta [e^{\rho(\cos\eta - \cos t)} e^{\int_\eta^t y_1(s)ds} + \arctan(\rho \sin t + y_1(t))] \sin t \, dt \\ &+ \int_\eta^\pi [e^{\rho(\cos\eta - \cos t)} e^{\int_\eta^t y_1(s)ds} + \arctan(\rho \sin t + y_1(t))] \sin t \, dt, \end{aligned}$$

for $\|y_1\|_\infty < \infty$. Now as $\rho \to \infty$,

$$\sum(\rho, y_1) \to \begin{cases} \infty, & \text{if } 0 \le \eta < \pi, \\ \pi, & \text{if } \eta = \pi, \end{cases}$$

and as $\rho \to -\infty$,

$$\sum(\rho, y_1) \to \begin{cases} \infty, & \text{if } 0 < \eta \le \pi, \\ -\pi, & \text{if } \eta = 0. \end{cases}$$

Consequently, when $\eta = 0$, it is sufficient that $-\pi < \int_0^\pi p(t) \sin t \, dt < \infty$, whereas when $\eta = \pi$, it is sufficient for $\pi < \int_0^\pi p(t) \sin t \, dt < \infty$.

Remark 2. Our approach leaves open the question of the existence of a necessary and sufficient condition for $\eta \neq 0$ or π.

EXAMPLE 4. Consider the problem

$$\begin{gathered} x''' + x' + \arctan(x) - \arctan(x') = p(t), \\ x'(0) = x'(\pi) = x(\eta) = 0, \end{gathered} \tag{8}$$

for $\eta \in [0, \pi]$. When $0 \le \eta < \pi$, problem (8) has a solution if $-\pi < \int_0^\pi p(t) \sin t \, dt < \pi$. When $\eta = \pi$, no solution for (8) is guaranteed. This example is a special case of the following corollary.

COROLLARY 1. *If g: $R \times R \to R$ is bounded with existing asymptotic limits $g(\pm\infty, \pm\infty)$ defined as*

$$g(\pm\infty, \pm\infty) = \lim_{\substack{u \to \pm\infty \\ v \to \pm\infty}} g(u, v),$$

then problem (1) *has a solution if $\int_0^\pi p(t) \sin t \, dt$ is between* $(1 - \cos\eta) g(-\infty, +\infty) + (1 + \cos\eta) g(+\infty, +\infty)$ *and* $(1 - \cos\eta) g(+\infty, -\infty) + (1 + \cos\eta) g(-\infty, -\infty)$, *whenever these two values are distinct.*

4. REMARKS

Problem (1) can be expressed in the form

$$\begin{gathered} ABx + g(x, Bx) = p(t), \\ x'(0) = x'(\pi) = x(\eta) = 0, \end{gathered} \tag{9}$$

where A and B are the operators corresponding to

$$Bx = x', \qquad x(\eta) = 0,$$

and

$$Ax = x'' + x, \qquad x(0) = x(\pi) = 0.$$

In this case, B is an invertible operator with inverse

$$(B^{-1}y)(t) = \int_{\eta}^{t} y(s)\, ds.$$

Problem (9) can be rewritten as

$$\begin{gathered} Ay + g\left(\int_{\eta}^{t} y(s)\, ds, y\right) = p(t), \\ y(0) = y(\pi) = 0, \end{gathered} \tag{10}$$

where $y = Bx$.

Analogously, we can study other problems of the form

$$\begin{gathered} Ay + g(B^{-1}, y) = p(t), \\ y(0) = y(\pi) = 0, \end{gathered} \tag{11}$$

where B is a suitable invertible operator on $D \subset C[0, \pi]$. For example, let B be the invertible operator $Bx = x'' + 2x$ with domain $D = \{x \in C^2[0, \pi] \mid x(0) = x(\pi) = 0\}$. The inverse of B is given by

$$(B^{-1}f)(t) = \int_{0}^{\pi} f(s)G(t, s)\, ds,$$

where G is the Green's function associated with $By = f$. With this choice of B, A as given before, and if one lets $x = B^{-1}y$, problem (11) is equivalent to

$$\begin{gathered} x^{(4)} + 3x'' + 2x + g(x, Bx) = p(t), \\ x''(0) = x''(\pi) = x(0) = x(\pi) = 0. \end{gathered} \tag{12}$$

Proceeding as before and using the fact that $B^{-1} \sin t = \sin t$, one can prove:

THEOREM 2. *Assume*

(i) $g\colon R \times R \to R$ *is a continuous function satisfying* $g(u, v) \geq 0$ *for all* (u, v) *in* $R \times R$.

(ii) $p \in C[0, \pi]$.

(iii) *There exist real numbers* $\rho_1 < \rho_2$ *and* $\delta > 0$ *such that*

$$\int_0^\pi g(\rho_1 \sin t + (B^{-1}y_1)(t), \rho_1 \sin t + y_1(t)) \sin t\, dt$$
$$< \int_0^\pi p(t) \sin t\, dt$$
$$< \int_0^\pi g(\rho_2 \sin t + (B^{-1}y_1)(t), \rho_2 \sin t + y_1(t)) \sin t\, dt,$$

for $\|y_1\|_\infty < \delta$.

(iv) *Moreover,* δ *further satisfies*

$$\frac{4K}{\pi}\left[\int_0^\pi |p(t)| \sin t\, dt\right] < \delta.$$

Then, problem (12) has a solution

$$x(t) = \rho \sin t + B^{-1}x_1,$$

where $\|x_1\|_\infty < K[|p_0| + (2/\pi) \int_0^\pi |p(t)| \sin t\, dt]$ and ρ is between ρ_1 and ρ_2.

As a consequence of Theorem 2, one can show that the nonlinear boundary value problem

$$\begin{aligned} x^{(4)} + 3x'' + 2x + e^x + e^{-x''} &= p(t), \\ x''(0) = x''(\pi) = x(0) = x(\pi) &= 0 \end{aligned} \tag{13}$$

has a solution provided

$$0 < \int_0^\pi p(t) \sin t\, dt < \infty.$$

References

1. A. R. Aftabizadeh, C. P. Gupta, and J. M. Xu, Existence and uniqueness theorems for three-point boundary value problems, *SIAM J. Math. Anal.* **20** (1989), 716–726.
2. A. R. Aftabizadeh and J. Wiener, Existence and uniqueness theorems for third order boundary value problems, *Rend. Sem. Mat. Univ. Padova* **75** (1986), 130–141.
3. R. P. Agarwal, Existence-uniqueness and iterative methods for third order boundary value problems, *J. Comput. Appl. Math.* **17** (1987), 271–289.
4. L. Cesari, Functional analysis, nonlinear differential equations and the alternative

method, *in* "Nonlinear Functional Analysis and Differential Equations" (L. Cesari, R. Kannan, and J. Schuur, Eds.), Dekker, New York, 1976.

5. M. Gregus, "Third Order Linear Differential Equations," Riedel, Dordrecht, 1987.
6. C. P. Gupta, On a third-order three-point boundary value problem at resonance, *Differential Integral Equations* **2** (1989), 1–12.
7. J. Henderson, Best interval lengths for boundary value problems for third order Lipschitz equations, *SIAM J. Math. Anal.* **18** (1987), 293–305.
8. R. Kannan and R. Ortega, Existence of solutions of $x'' + x + g(x) = p(t), x(0) = 0 = x(\pi)$, *Proc. Amer. Math. Soc.* **96** (1986), 67–70.
9. D. Krajcinovic, Sandwich beam analysis, *J. Appl. Mech.* **39** (1972), 773–778.
10. J. Mawhin, Topological degree methods in nonlinear boundary value problems, *in* "CBMS Regional Conf. Ser. in Math., No. 40," Amer. Math. Soc., Providence, RI, 1979.
11. K. N. Murty and B. D. C. N. Prasad, Three-point boundary value problems, existence and uniqueness, *Yokohama Math. J.* **29** (1981), 101–105.
12. K. N. Murty and B. D. C. N. Prasad, Application of Liapunov theory to three-point boundary value problems, *J. Math. Phys. Sci.* **19** (1985), 225–234.
13. D. J. O'Reagan, Topological transversality: Applications to third order boundary value problems, *SIAM J. Math. Anal.* **19** (1987), 630–641.